Spectral methods for PDEs

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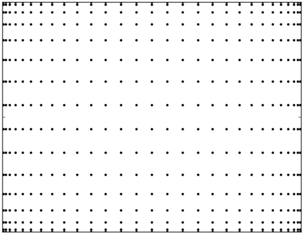

November 30th 2017, Vienna, Austria

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Generic Evolution Equation

$$\boxed{\alpha \frac{\partial u}{\partial t} = \mathcal{M}(u)}$$

$\alpha \partial_t u$ = non-steady term

$\mathcal{M}(u)$ = differential operator (diffusion, advection, reaction, etc...)

Solution as Linear Combination of Trial Functions

$$\boxed{u(\boldsymbol{x}, t) = \sum_{i,j,k=0}^{\infty} a_{i,j,k}(t)\phi_{i,j,k}(\boldsymbol{x})}$$

- Tensor product: $\phi_{i,j,k}(\boldsymbol{x}) = \phi_i(x)\psi_j(y)\gamma_k(z)$
- $\phi_i,\ \psi_j,\ \gamma_k$ basis of orthogonal polynomials

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Orthogonal Polynomials

$$\int_{x_0}^{x_1} \phi_k(x)\overline{\phi_l(x)}w(x)dx = c_k\delta_{kl}$$

- $[x_0 \, , x_1]$ = prototypical domain
- $w(x) =$ weight internal product
- $\delta_{kl} =$ Kronecker delta function
- Fourier modes/eigenfunctions of the Sturm-Liouville problem

Fourier:

$\phi_k(x) = e^{ikx}$

$[x_0 \, , x_1] = [0 \, , 2\pi]$

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Legendre:

$$\phi_{k+1}(x) = L_{k+1}(x) = \frac{2k+1}{k+1}xL_k(x) - \frac{k}{k+1}L_{k-1}(x),\ L_0 = 1,\ L_1 = x$$

$$[x_0\ , x_1] = [-1\ , 1],\ \ w(x) = 1$$

$$c_k = (k+1/2)^{-1}$$

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Chebyshev:

$$\phi_k(x) = T_k(x) = cos(k\ cos^{-1}x), \quad k = 0,\ 1,\ \ldots$$

$$[x_0\ , x_1] = [-1\ , 1], \ \ w(x) = (1 - x^2)^{-1/2}$$

$$c_k = \pi \ \text{ if } \ k = 0, \ \ c_k = \pi/2 \ \text{ if } \ k \geq 1$$

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- $\phi_i,\ \psi_j,\ \gamma_k$ basis of orthogonal polynomials
- $a_{i,j,k}(t)$ modes of the linear expansion

Nodal Approach

$$u(\boldsymbol{x},t) = \sum_{i,j,k=0}^{\infty} u_{i,j,k}(t)\delta_{i,j,k}(\boldsymbol{x})$$

- Lagrangian functions: $\delta_{i,j,k}(\boldsymbol{x}) = \delta_i(x)\delta_j(y)\delta_k(z)$
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$$\delta_l(x) = \prod_{k \neq l, 0 \leq k, l \leq N} \frac{x - x_k}{x_l - x_k}$$

- $\delta_l(x_k) = \delta_{lk}$: Kronecker delta
- x_k = roots of Legendre, Chebyshev, ... polynomials
- Fourier modes: x_k are evenly spaced

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- Boundary conditions on one boundary
  - → Legendre– and Chebyshev–Gauss–Randau (polar coordinates)
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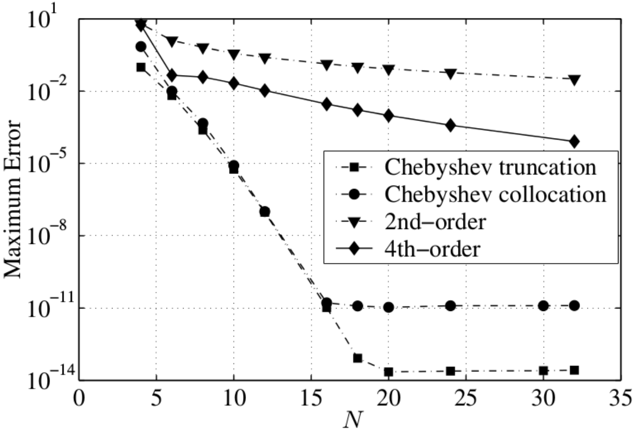
Convergence Curves: Diffusion Equation 1-D



Advection Equation

$$\frac{\partial u}{\partial t} + a(x)\frac{\partial u}{\partial x} = 0 \quad , \quad x \in [0, 2\pi] \text{ , periodic BC}$$

$$a(x) = \sin^2(x-1) + 1/5 \text{ ,} \quad u(x, t=0) = e^{-100(x-1)^2}$$

Discretization

- Leap-Frog finite difference method for time
- Fourier modes for space

$$\frac{\boldsymbol{u}_N^{n+1} - \boldsymbol{u}_N^{n-1}}{2\Delta t} = -a(\boldsymbol{x}_N)[D_N]\boldsymbol{u}_N^n$$

$\boldsymbol{u}_N^n =$ discrete solution on N nodes in space at time n

$[D_N] =$ derivation matrix in space on Fourier nodes (Toeplitz)

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$$\frac{\partial u}{\partial t} = \sigma \frac{\partial^2 u}{\partial x^2} \quad , \quad x \in [-1, 1] \ , u(\pm 1) = \pm 1/2$$
$$\sigma = 0.01 \ , \quad u(x, t = 0) = 1/2 \sin(\pi x/2)$$

Discretization

- Explicit Euler finite difference method for time
- Chebyshev nodes for space

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Poisson Equation

$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = f(x,y) \quad , \quad (x,y) \in [-1,1]^2 \ , u_{BC} = 0$$

$$f(x,y) = 2y\sin(x)$$

Discretization

- Chebyshev nodes for space
- Kronecker product for 2-D matrices

$$([D_{xN}]^2_{2D} + [D_{yN}]^2_{2D})\boldsymbol{u}_N = f(\boldsymbol{x}_N, \boldsymbol{y}_N)$$

$\boldsymbol{u}^n_N =$ discrete solution on $N = N_x \times N_y$ nodes in 2-D space

$[D_{xN}]^2_{2D} = [I]_{1D} \otimes [D_x]^2_{1D}, \quad [D_{yN}]^2_{2D} = [D_y]^2_{1D} \otimes [I]_{1D}$

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